Molecule Distribution in the Detection Region

Lets assume that we start with all molecules exiting the beam source at $\ell = 0$ and $t = 0$ with random longitudinal velocity v which is guassianly distributed with mean $\bar{v}$, and standard deviation σ. This probability distribution is given by:

$$P_v(v) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(v-\bar{v})^2}{2\sigma^2}\right)$$

Now, we would like to know the probability distribution for the arrival time $T = L/v$ of the molecules in the detection region which is a distance L away from the beam source.

This is given by:

$$\begin{aligned} P_T(T) =& P_v\left(v = \frac{L}{T}\right)\left|\frac{\partial v}{\partial T}\right| \\ =& \frac{1}{\sqrt{2\pi\sigma^2}}\frac{L}{T^2}\exp\left(-\frac{\left(\frac{L}{T}-\bar{v}\right)^2}{2\sigma^2}\right) \end{aligned}$$

where $\partial v/\partial T$ is the Jacobian determinant of the coordinate transformation that we did from velocity space to arrival time space.

This probability distribution is normalized and is asymetric in T. This problem becomes harder when we include the probability distribution $P(t, v)\, dt\, dv$ for random time t exiting the cell and random longitudinal velocity v. In this case, the arrival time is $T = t + L/v$, and the final velocity is the same as the initial velocity, $V = v$. So in this case, the probability distribution for velocity and arrival time is given by:

$$P_1(T, V) = P_0\left(t = T - \frac{L}{V}, v = V\right)\left|\det\frac{\partial(T, V)}{\partial(t, v)}\right|$$

For this coordinate transformation, the jacobian determinant is 1, $\left|\det\frac{\partial(T,V)}{\partial(t,v)}\right| = 1$. Then, if we want just the probability distribution over arrival time, then we evaluate:

$$P_T(T) = \int dV\; P_1(T, V)$$

The first example implicitly had a delta function over cell exiting time:

$$P_0(t, v) = \delta(t)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(v-\bar{v})^2}{2\sigma^2}\right)$$

and so we obtain:

$$\begin{aligned} P_T(T) =& \frac{1}{\sqrt{2\pi\sigma^2}}\int dV\;\delta\left(T - \frac{L}{V}\right)\;\exp\left(-\frac{(v-\bar{v})^2}{2\sigma^2}\right) \\ =& \frac{1}{\sqrt{2\pi\sigma^2}}\frac{L}{T^2}\exp\left(-\frac{\left(\frac{L}{T}-\bar{v}\right)^2}{2\sigma^2}\right) \end{aligned}$$

which is the same as that which we obtained using the other method. Now, we can generalize to include a gaussian distribution for cell departure times:

$$P_0(t, v) = \frac{1}{\sqrt{(2\pi)^2\sigma_v^2\sigma_t^2}}\exp\left(-\frac{(v-\bar{v})^2}{2\sigma_v^2} - \frac{t^2}{2\sigma_t^2}\right)$$

The integral required to evaluate $P_T(T)$ is difficult to do analytically in this case, but it can be done numerically. The introduction of a nonzero σ_t will lead to an increased variance in $P_T(T)$.

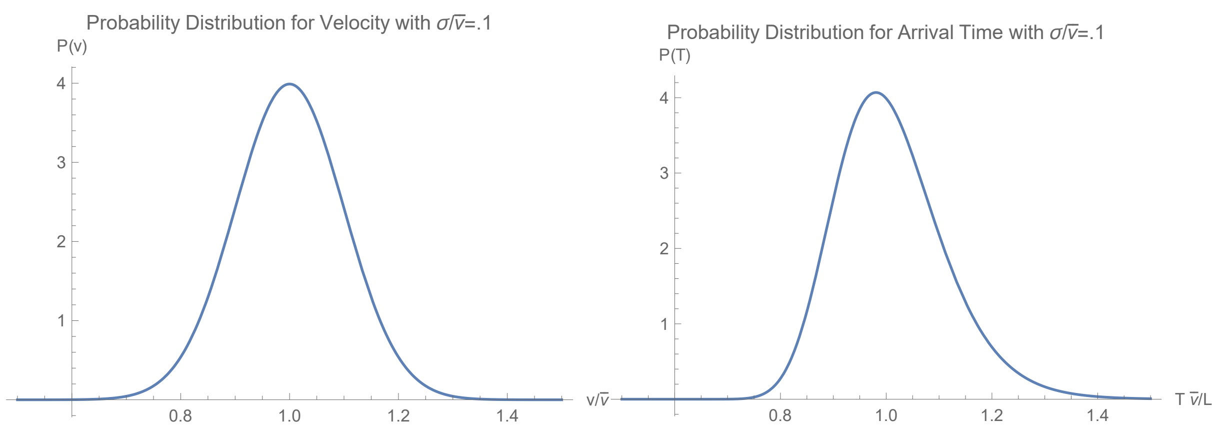

Figure 1: Left: Symmetric distribution $P_v(v)$. Right: Asymmetric distribution $P_T(T)$.